S.-T. Yau College Student Mathematics Contest

Applied Mathematics (Group)2015

1. Find the following generalized eigen value λ and the eigen vector $(x_1, x_2, \cdots, x_n)^T$, such that

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}_{n\times n} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

2. Let A be a positive definite matrix. Consider the quadratic function $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Suppose the eigenvalues of A are given by

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, k = \frac{\lambda_n}{\lambda_1}.$$

Let the sequence $\{\mathbf{x}_k\}$ be generated by the steepest descent method:

$$x_{k+1} = x_k - \alpha_k \nabla g(x_k), \quad k = 0, 1, 2, \cdots,$$

where α_k is selected such that

$$g(x_k - \alpha_k \nabla g(x_k)) = \min_{\alpha \ge 0} g(x_k - \alpha \nabla g(x_k)).$$

Prove that

$$g(\mathbf{x}_{k+1}) \le \left(\frac{k-1}{k+1}\right)^2 g(\mathbf{x}_k) \le \left(\frac{k-1}{k+1}\right)^{2(k+1)} g(\mathbf{x}_0).$$

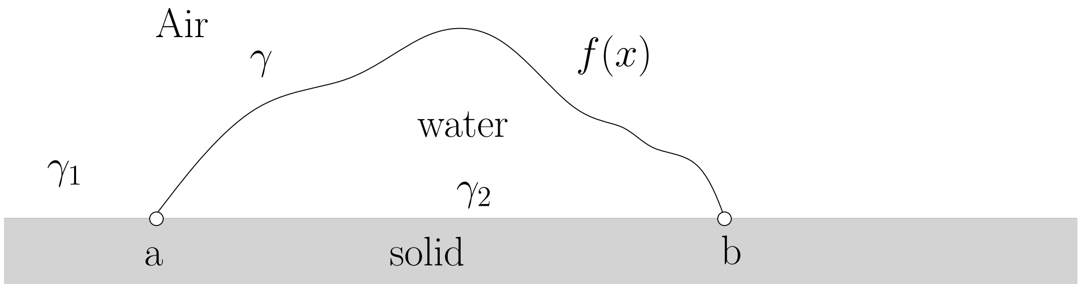

Figure 1: Equilibrium shape of water droplet.

3. Consider a water droplet on a solid surface as shown in Figure 1. The air-water surface tension is γ, the air-solid surface tension is γ_1, the water-solid surface tension is γ_2. The interface energy = surface tension $\times$ length of the interface. The equilibrium state should minimizes the total interface energy. (neglecting gravity)

   Formulate a variational problem to show that the equilibrium droplet should have circular shape (i.e. the curvature of the curve $f(x)$ is a constant).